

Is $\theta_{13}^{\text{PMNS}}$ correlated with $\theta_{23}^{\text{PMNS}}$ or not?

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Abstract

By postulating the relation $\theta_{23} \simeq 45^\circ + \eta\theta_{13}$, we seek for preferable correction terms to the charged lepton and neutrino mass matrices starting from the Tri-Bi-Maximal mixing and discuss their origins. The global fits of neutrino oscillation parameters favor $\eta = \pm 1/\sqrt{2}$; it ends up with the relation found by Edy, Frampton and Matsuzaki some years ago in the context of a T' flavor model. In contrast, the results of the ν_μ disappearance mode reported by the T2K and Super-Kamiokande collaborations seem to prefer $\eta = 0$, which turns out an almost maximal θ_{23} . We derive a general condition for obtaining $\theta_{23} \simeq 45^\circ + \eta\theta_{13}$. The condition is complicated by the neutrino masses and CP violating phases, so that we focus on the cases of a specific neutrino mass spectrum and/or characteristic CP phases. Under such simplified environments, we find several correction terms that take very simple forms. It is also found that the obtained correction terms indeed arise from the spontaneous breaking of a flavor symmetry or one-loop radiative corrections.

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I. INTRODUCTION

Now that θ_{13} of the PMNS mixing matrix has been measured very precisely by the reactor [1] and long-baseline [2] neutrino oscillation experiments, it may be said that we have succeeded in acquiring a clear picture of the neutrino mixing pattern. It is the Daya-Bay experiment that holds the record of a precise determination of θ_{13} : the vanishing θ_{13} is now excluded at the level of 7.7σ standard deviations with an unexpectedly large central value of $\theta_{13}^{\text{DB}} \simeq 8.7^\circ$ [3]. Such a large θ_{13} would offer a great opportunity for us to explore the ordering of neutrino masses, the octant of θ_{23} and leptonic CP violation.

From a theoretical side, the discovery of a large θ_{13} might make people depressed since that might signal the end of a paradigm of the Tri-Bi-Maximal (TBM) mixing [4], which predicts $\theta_{13} = 0^\circ$. Nevertheless, the TBM mixing may still be available as a leading order one in the presence of small corrections. In fact, various ways to complement the TBM mixing by perturbing the neutrino sector [5, 6], the charged lepton sector [7] or both [8], have recently been proposed. In the present study, we take the same stance while paying special attention to correlation between θ_{13} and θ_{23} .

Within a framework of the TBM mixing plus small corrections, the deviations of θ_{13} and θ_{23} from their TBM values are given by the same parameters as we shall show in Eqs. (10) and (11); thus it is likely that they are correlated with each other. Indeed, it often happens in some flavor models [9]. Given this fact, it is very intriguing to recall the relation $\theta_{13} = \sqrt{2}|45^\circ - \theta_{23}|$ found by Edy, Frampton and Matsuzaki in the context of a T' flavor symmetry [10] (see also Ref. [11] for earlier works). We will hereafter refer to this relation as the EFM relation. Interestingly, the EFM relation is now in excellent agreement with the global fits of neutrino oscillation parameters as demonstrated in Tab. I. Such a simple relation is expected to help people have an insight for model-building, although it should not be exact¹.

In contrast, the results of the ν_μ disappearance mode reported by the T2K [14] and Super-Kamiokande [15] collaborations seem to still favor the maximal θ_{23} , indicating that only θ_{13} departs from the TBM value independently of θ_{23} . If this is the case, we would be forced to explain the stability of θ_{23} while inducing an appreciable deviation for θ_{13} .

¹ At least, we do not find any underlying symmetry which guarantees the EFM relation. Thus, it should be affected by the running of RGEs for instance.

Data	$\theta_{23}^{\text{best}}$	$\theta_{13}^{\text{best}}$	θ_{13}^{EFM}
Ref. [12]	$38.4^\circ (38.7^\circ)$	$8.9^\circ (9.0^\circ)$	$9.3^\circ (8.9^\circ)$
Ref. [13]	$51.5^\circ (50.8^\circ)$	$9.0^\circ (9.1^\circ)$	$9.2^\circ (8.2^\circ)$

TABLE I. A comparison of the EFM predictions with the global fits of neutrino oscillation parameters in the case of normal (inverted) mass ordering: $\theta_{13}^{\text{best}}$ and $\theta_{23}^{\text{best}}$ are the best fit values from Refs. [12] and [13], while θ_{13}^{EFM} is a prediction of the EFM relation when employing $\theta_{23}^{\text{best}}$.

This requirement could be useful information when complementing the TBM mixing.

In view of these thoughts, we seek for preferable correction terms to the charged lepton and neutrino mass matrices with a guide of

$$\theta_{23} \simeq 45^\circ + \eta \theta_{13}, \quad (1)$$

where $\eta = \pm 1/\sqrt{2}$ ends up with the EFM relation, while $\eta = 0$ corresponds to the case of an almost maximal θ_{23} . Note that we use \simeq in Eq. (1), because we do not believe this relation holds exactly and also some approximations are used in our calculations. After showing our definitions of mass and mixing matrices in Sec. II, we derive a general condition for deriving Eq. (1) in Sec. III. The obtained condition is somewhat complicated by the neutrino masses and CP violating phases, and it does not seem to provide a simple correction term realizable in flavor models. Thus, in Sec. IV, we restrict ourselves to the cases of a specific neutrino mass spectrum and/or characteristic CP violating phases and show several examples yielding Eq. (1). In Sec. V, we develop two possible ways to realize the correction terms obtained in Sec. IV by means of a flavor symmetry or radiative corrections. We summarize our discussions in Sec. VI.

II. DEFINITIONS

We begin with the SM Lagrangian augmented by an effective Majorana-mass-term of the left-handed neutrinos:

$$\mathcal{L} = \mathcal{L}_{\text{SM}} + \nu_L^\dagger M_\nu \nu_L^* + h.c. , \quad (2)$$

and consider circumstances under which θ_{13} and θ_{23} are enforced to be 0° and 45° , respectively, by an underlying flavor physics at leading order. Until Sec. IV, we remain

θ_{12} arbitrary in order to keep our discussions as general as possible. The leading-order PMNS matrix is defined as $U^0 = (U_\ell^0)^\dagger U_\nu^0 = V^0 P^0$, where U_ℓ^0 and U_ν^0 stand for the leading-order diagonalizing-matrices of the charged leptons and neutrinos, respectively, $P^0 = \text{Diag}(e^{i\alpha/2}, e^{i\beta/2}, 1)$ and

$$V^0 = \begin{pmatrix} c_{12}^0 & s_{12}^0 & 0 \\ -s_{12}^0/\sqrt{2} & c_{12}^0/\sqrt{2} & -1/\sqrt{2} \\ -s_{12}^0/\sqrt{2} & c_{12}^0/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}. \quad (3)$$

α and β are Majorana CP violating phases, and s_{12}^0 (or c_{12}^0) denotes $\sin \theta_{12}^0$ (or $\cos \theta_{12}^0$). We divide mass matrices into leading and correction terms in such a way that

$$M_\ell = M_\ell^0 + \delta M_\ell, \quad M_\nu = M_\nu^0 + \delta M_\nu,$$

where M_ℓ is a mass matrix of the charged leptons. In this definition, M_ℓ^0 and M_ν^0 are postulated to be diagonalized by U_ℓ^0 and U_ν^0 , respectively, and generate U^0 , whereas δM_ℓ and δM_ν give rise to slight deviations from U^0 . In what follows, we proceed with our analysis based on a premise that the correction terms are sufficiently smaller than the leading terms. Approximating the diagonalization of M_ℓ as

$$(U_\ell^0 + \delta U_\ell)^\dagger M_\ell M_\ell^\dagger (U_\ell^0 + \delta U_\ell) \simeq \text{Diag}(m_e^2, m_\mu^2, m_\tau^2),$$

we move on to the diagonal basis of M_ℓ and redefine M_ν as follows:

$$\begin{aligned} M_\nu &\rightarrow \bar{M}_\nu = (U_\ell^0 + \delta U_\ell)^\dagger (M_\nu^0 + \delta M_\nu) (U_\ell^0 + \delta U_\ell)^* \\ &\simeq (U_\ell^0)^\dagger M_\nu^0 (U_\ell^0)^* + \delta U_\ell^\dagger M_\nu^0 (U_\ell^0)^* + (U_\ell^0)^\dagger M_\nu^0 \delta U_\ell^* + (U_\ell^0)^\dagger \delta M_\nu (U_\ell^0)^* \\ &\equiv \bar{M}_\nu^0 + \delta \bar{M}_\nu, \end{aligned} \quad (4)$$

where $\bar{M}_\nu^0 = (U_\ell^0)^\dagger M_\nu^0 (U_\ell^0)^*$ and we have dropped several terms in the second line. We stress that $\delta \bar{M}_\nu$ includes corrections stemming from both the charged lepton and neutrino sectors.

One of purposes of this study is to derive $\delta \bar{M}_\nu$ leading to $\theta_{23} \simeq 45^\circ + \eta \theta_{13}$; so we go into more detail about $\delta \bar{M}_\nu$ as well as \bar{M}_ν^0 . \bar{M}_ν^0 is diagonalized by V^0 , i.e. $(V^0)^\dagger \bar{M}_\nu^0 (V^0)^* = \text{Diag}(m_1^0 e^{i\alpha}, m_2^0 e^{i\beta}, m_3^0) \equiv \mathcal{D}_\nu^0$, and expressed in terms of the leading-order mixing angles

θ_{ij}^0 and neutrino masses m_i^0 :

$$\begin{aligned}
(\bar{M}_\nu^0)_{11} &= m_1^0 e^{i\alpha} (c_{12}^0)^2 + m_2^0 e^{i\beta} (s_{12}^0)^2, \\
(\bar{M}_\nu^0)_{22} &= (M_\nu^0)_{33} = \frac{1}{2} [m_1^0 e^{i\alpha} (s_{12}^0)^2 + m_2^0 e^{i\beta} (c_{12}^0)^2 + m_3^0], \\
(\bar{M}_\nu^0)_{12} &= (M_\nu^0)_{13} = \frac{1}{\sqrt{2}} (-m_1^0 e^{i\alpha} + m_2^0 e^{i\beta}) s_{12}^0 c_{12}^0, \\
(\bar{M}_\nu^0)_{23} &= \frac{1}{2} [m_1^0 e^{i\alpha} (s_{12}^0)^2 + m_2^0 e^{i\beta} (c_{12}^0)^2 - m_3^0],
\end{aligned} \tag{5}$$

where m_i^0 are real and positive. Meanwhile, $\delta\bar{M}_\nu$ is parametrized by six complex parameters, such that

$$\delta\bar{M}_\nu = \begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix},$$

yet only three of $a \cdots f$ are independent parameters of the system. In order to see that, let us consider the diagonal basis of \bar{M}_ν^0 , namely the basis of $(V^0)^\dagger [\bar{M}_\nu^0 + \delta\bar{M}_\nu] (V^0)^* \equiv \mathcal{D}_\nu^0 + \delta\mathcal{D}_\nu$, and define $\delta\mathcal{D}_\nu$ as

$$\delta\mathcal{D}_\nu = \begin{pmatrix} 0 & X & Y \\ X & 0 & Z \\ Y & Z & 0 \end{pmatrix}, \tag{6}$$

where X , Y and Z are complex parameters, and the diagonal entries are omitted since they can be absorbed into \mathcal{D}_ν^0 . In turn, going back to the basis of $\bar{M}_\nu^0 + \delta\bar{M}_\nu$, one finds

$$\begin{aligned}
a &= 2s_{12}^0 c_{12}^0 X, \\
b &= -s_{12}^0 c_{12}^0 X + (s_{12}^0 Y - c_{12}^0 Z), \\
c &= -s_{12}^0 c_{12}^0 X - (s_{12}^0 Y - c_{12}^0 Z), \\
d &= \frac{1}{\sqrt{2}} \{ [(c_{12}^0)^2 - (s_{12}^0)^2] X - (c_{12}^0 Y + s_{12}^0 Z) \}, \\
e &= \frac{1}{\sqrt{2}} \{ [(c_{12}^0)^2 - (s_{12}^0)^2] X + (c_{12}^0 Y + s_{12}^0 Z) \}, \\
f &= -s_{12}^0 c_{12}^0 X.
\end{aligned} \tag{7}$$

Thus, throughout this paper, we will express $\delta\bar{M}_\nu$ in terms of X , Y and Z .

III. RELATING θ_{13} WITH θ_{23}

By considering $\bar{M}_\nu \bar{M}_\nu^{\dagger 2}$:

$$\bar{M}_\nu \bar{M}_\nu^\dagger \simeq \bar{M}_\nu^0 (\bar{M}_\nu^0)^\dagger + \bar{M}_\nu^0 (\delta \bar{M}_\nu)^\dagger + \delta \bar{M}_\nu (\bar{M}_\nu^0)^\dagger,$$

and by regarding the second and third terms as small perturbations to $\bar{M}_\nu^0 (\bar{M}_\nu^0)^\dagger$, V_{ij}^0 are perturbed to be

$$\begin{aligned} V_{e1} &= c_{12}^0 - \frac{m_1^0 e^{-i\alpha} X + m_2^0 e^{i\beta} X^*}{(m_2^0)^2 - (m_1^0)^2} s_{12}^0, \\ V_{e2} &= s_{12}^0 + \frac{m_1^0 e^{i\alpha} X^* + m_2^0 e^{-i\beta} X}{(m_2^0)^2 - (m_1^0)^2} c_{12}^0, \\ V_{e3} &= \frac{m_1^0 e^{i\alpha} Y^* + m_3^0 Y}{(m_3^0)^2 - (m_1^0)^2} c_{12}^0 + \frac{m_2^0 e^{i\beta} Z^* + m_3^0 Z}{(m_3^0)^2 - (m_2^0)^2} s_{12}^0, \\ V_{\mu 3} &= -\frac{1}{\sqrt{2}} \left[1 + \frac{m_1^0 e^{i\alpha} Y^* + m_3^0 Y}{(m_3^0)^2 - (m_1^0)^2} s_{12}^0 - \frac{m_2^0 e^{i\beta} Z^* + m_3^0 Z}{(m_3^0)^2 - (m_2^0)^2} c_{12}^0 \right], \\ V_{\tau 3} &= \frac{1}{\sqrt{2}} \left[1 - \left\{ \frac{m_1^0 e^{i\alpha} Y^* + m_3^0 Y}{(m_3^0)^2 - (m_1^0)^2} s_{12}^0 - \frac{m_2^0 e^{i\beta} Z^* + m_3^0 Z}{(m_3^0)^2 - (m_2^0)^2} c_{12}^0 \right\} \right], \end{aligned} \quad (8)$$

whereas there are no corrections to the eigenvalues upto the first order expansion. From these elements, the perturbed $\tan \theta_{12}$, $\tan \theta_{23}$ and $\sin \theta_{13}$ are found to be

$$\tan \theta_{12} = \left| \frac{V_{e2}}{V_{e1}} \right| \simeq t_{12}^0 \left[1 + \frac{1}{s_{12}^0 c_{12}^0} \text{Re} \frac{m_1^0 e^{i\alpha} X^* + m_2^0 e^{-i\beta} X}{(m_2^0)^2 - (m_1^0)^2} \right], \quad (9)$$

$$\tan \theta_{23} = \left| \frac{V_{\mu 3}}{V_{\tau 3}} \right| \simeq 1 + 2 \text{Re} \left[\frac{m_1^0 e^{i\alpha} Y^* + m_3^0 Y}{(m_3^0)^2 - (m_1^0)^2} s_{12}^0 - \frac{m_2^0 e^{i\beta} Z^* + m_3^0 Z}{(m_3^0)^2 - (m_2^0)^2} c_{12}^0 \right], \quad (10)$$

$$\sin \theta_{13} = |V_{e3}| = \left| \frac{m_1^0 e^{i\alpha} Y^* + m_3^0 Y}{(m_3^0)^2 - (m_1^0)^2} c_{12}^0 + \frac{m_2^0 e^{i\beta} Z^* + m_3^0 Z}{(m_3^0)^2 - (m_2^0)^2} s_{12}^0 \right|, \quad (11)$$

respectively, and it can be seen that X is mainly responsible for the deviations of θ_{12} while the deviations of θ_{23} and θ_{13} are controlled by Y and Z .

As for θ_{13} and θ_{23} , we are particularly interested in the relation $\theta_{23} \simeq 45^\circ + \eta \theta_{13}$ with

$$\eta = \begin{cases} \pm 1/\sqrt{2} \\ 0 \end{cases}.$$

In view of $\cos \theta_{13} \simeq 1$, this relation may be translated into

$$\sin \theta_{23} = \sin[45^\circ + \eta \theta_{13}] \simeq \frac{1}{\sqrt{2}} (1 + \eta \sin \theta_{13}),$$

² In the first paper of Ref. [6], a perturbation method is also adopted, but for \bar{M}_ν in view of $V_{ij}^* \simeq V_{ij}$.

As we shall show in Sec. IV-B, we will arrive at the same conclusion.

leading to

$$|V_{\mu 3}| \simeq \frac{1}{\sqrt{2}} [1 + \eta |V_{e3}|].$$

Then, by substituting V_{e3} and $V_{\mu 3}$ derived above, one arrives at

$$\begin{aligned} \eta \left| \frac{m_1^0 e^{i\alpha} Y^* + m_3^0 Y}{(m_3^0)^2 - (m_1^0)^2} c_{12}^0 + \frac{m_2^0 e^{i\beta} Z^* + m_3^0 Z}{(m_3^0)^2 - (m_2^0)^2} s_{12}^0 \right| \\ = \text{Re} \left[\frac{m_1^0 e^{i\alpha} Y^* + m_3^0 Y}{(m_3^0)^2 - (m_1^0)^2} s_{12}^0 - \frac{m_2^0 e^{i\beta} Z^* + m_3^0 Z}{(m_3^0)^2 - (m_2^0)^2} c_{12}^0 \right]. \end{aligned} \quad (12)$$

This is the condition for obtaining $\theta_{23} \simeq 45^\circ + \eta \theta_{13}$ and one of our main results in this study. From this condition, one may be able to extract a relational expression between Y and Z , which enables us to know the form of $\delta \bar{M}_\nu$. Unfortunately, without any simplification, it does not seem the condition provide a simple $\delta \bar{M}_\nu$ realizable in flavor models. Hence, in the next section, we will deal with the condition under simplified environments, e.g. vanishing Majorana phases, quasi-degenerate neutrino masses, etc..

IV. DEVIATIONS FROM TBM MIXING

From now on, we concentrate on the case of $s_{12}^0 = 1/\sqrt{3}$; namely V^0 in Eq. (3) takes the form of the TBM mixing pattern. Moreover, we will postulate $(m_3^0)^2 - (m_1^0)^2 = (m_3^0)^2 - (m_2^0)^2$ in Eq. (12). As mentioned just below Eq. (8), the eigenvalues are only moderately corrected; it may be a good approximation to identify m_i^0 with the real neutrino masses m_i . Therefore, in the light of $\Delta m_{12}^2 \ll \Delta m_{23}^2$, postulating $(m_3^0)^2 - (m_1^0)^2 = (m_3^0)^2 - (m_2^0)^2$ is expected to be reasonable. Even with this simplification, however, Eq. (12) remains complicated. Thus, in the following subsections, we will restrict ourselves to the cases of a specific neutrino mass spectrum and/or characteristic CP violating phases. For the purpose of reference, we depict the running of neutrino masses as a function of the lightest neutrino mass in Fig. 1. In all of our numerical calculations, the following best fit values and 1σ errors from Ref. [12] are used:

$$\begin{aligned} \Delta m_{23}^2 &= \begin{cases} (2.43_{-0.10}^{+0.06}) \times 10^{-3} & \text{for Normal ordering} \\ (2.42_{-0.11}^{+0.07}) \times 10^{-3} & \text{for Inverted ordering} \end{cases}, \\ \Delta m_{12}^2 &= (7.54_{-0.22}^{+0.26}) \times 10^{-5}, \quad \sin^2 \theta_{12} = 0.307_{-0.016}^{+0.018}. \end{aligned} \quad (13)$$

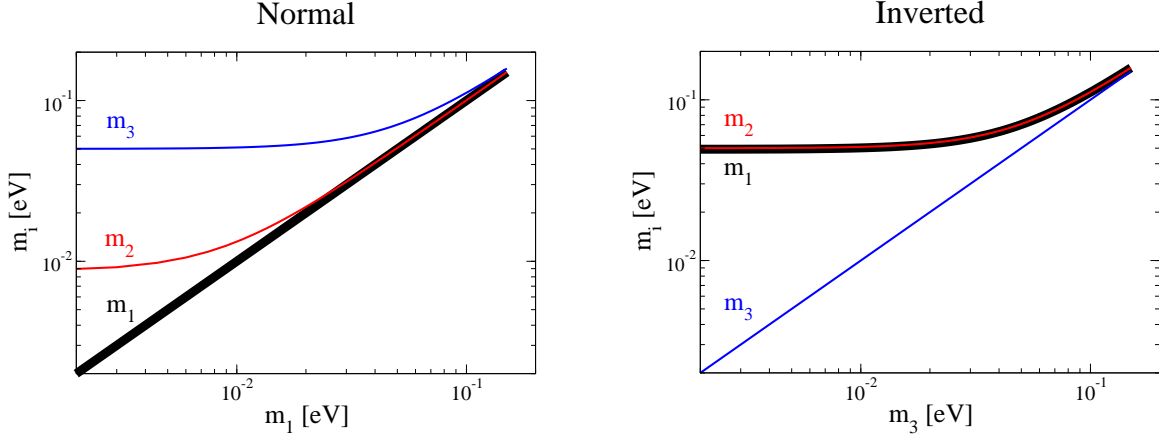


FIG. 1. The masses of neutrinos as a function of the lightest neutrino mass for the normal (left panel) and inverted (right panel) ordering cases. The best fit values are input for Δm_{12}^2 and Δm_{23}^2 . The black, red and blue curves correspond to m_1 , m_2 and m_3 , respectively.

A. $\eta = \pm 1/\sqrt{2}$

- Case A-I: $Y = Y^*$, $Z = Z^*$ and $\alpha = \beta = 0$ or π .

Assuming that Y and Z are real parameters and that $\alpha = \beta = 0$ or π , Eq. (12) is solved to be

$$Z \simeq \frac{\kappa - \eta\sqrt{2}}{\eta + \sqrt{2}\kappa} Y \quad (14)$$

with $[Y - \sqrt{2}Z]/\eta > 0$ (< 0) for the case of normal (inverted³) mass ordering. In deriving Eq. (14), we have assumed $\pm m_1^0 + m_3^0 = \pm m_2^0 + m_3^0$. A validity of this approximation is subject to a neutrino mass spectrum. When $m_1 \simeq m_2$, it should be applicable. Furthermore, the approximation appears to be valid in a mass region where m_1 is much smaller than m_2 in the case of normal ordering because $m_3 \gg m_{1,2}$ in that region. Nevertheless, as we shall see below, the difference between m_1 and m_2 gives rise to slight errors making the EFM relation somewhat hazed. $\kappa = \pm 1$ stems from a sign ambiguity originating in the left-handed side of Eq. (12) and provides us with two possible solutions. For $\eta = 1/\sqrt{2}$ ($-1/\sqrt{2}$), Eq. (14) results

³ Note that a minus sign appears in the right-handed side of Eq. (12) in the case of inverted mass ordering.

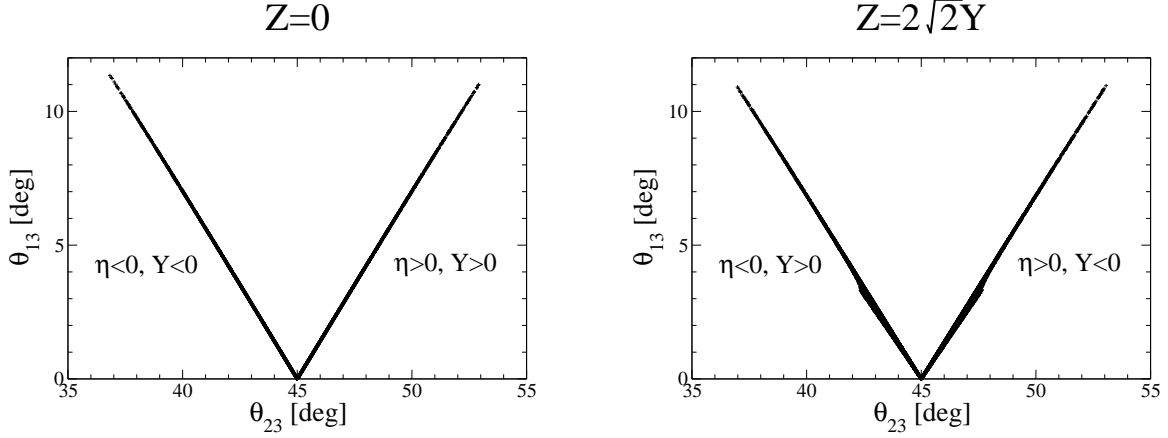


FIG. 2. θ_{13} as a function of θ_{23} for $Z = 0$ (left panel) and $Z = 2\sqrt{2}Y$ (right panel) in the case of normal mass ordering for Case A-I. Δm_{12}^2 , Δm_{23}^2 and θ_{12} are restricted to be within the 1σ bounds.

in $Z = 0$ with $Y > 0$ ($Y < 0$) and $Z = 2\sqrt{2}Y$ with $Y < 0$ ($Y > 0$) in the case of normal mass ordering, yielding

$$\delta \bar{M}_\nu = \frac{X}{3\sqrt{2}} \begin{pmatrix} 4 & 1 & 1 \\ 1 & -2 & -2 \\ 1 & -2 & -2 \end{pmatrix} + \frac{Y}{\sqrt{3}} \begin{pmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \quad \text{for } Z = 0, \quad (15)$$

$$\delta \bar{M}_\nu = \frac{X}{3\sqrt{2}} \begin{pmatrix} 4 & 1 & 1 \\ 1 & -2 & -2 \\ 1 & -2 & -2 \end{pmatrix} + \sqrt{3}Y \begin{pmatrix} 0 & -1 & 1 \\ -1 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad \text{for } Z = 2\sqrt{2}Y, \quad (16)$$

where X is responsible for the deviations of θ_{12} and in general complex. The sign of Y is flipped for the inverted ordering case. Note that in the case of $Z = 0$, the dependence of Eq. (12) on the Majorana phase β disappears; therefore, β can actually take any value in this case.

In Fig. 2, we numerically diagonalize the neutrino mass matrix Eq. (4) with Eqs. (15) and (16) and compute θ_{13} as a function of θ_{23} for the case of normal mass ordering. m_1^0 and $|Y|$ are varied within $0 \sim 0.1$ eV and $0 \sim 0.003$ eV ($0 \sim 0.001$ eV) for $Z = 0$ ($Z = 2\sqrt{2}Y$), while m_2^0 , m_3^0 and X are suitably tuned to be consistent with the 1σ constraints of Δm_{12}^2 , Δm_{23}^2 and θ_{12} . In the case of $Z = 2\sqrt{2}Y$, the EFM relation is slightly hazed due to the errors of an approximation of $\pm m_1^0 + m_3^0 =$

$\pm m_2^0 + m_3^0$ when m_1^0 is small. In contrast, the $Z = 0$ case does not suffer from the errors since it does not rely on the approximation. Almost the same figures are obtained for the inverted ordering case, but such a haze does not show up since m_1 is always close to m_2 .

- Case A-II: $\alpha = \beta = 0$ in the case of quasi-degenerate neutrino mass spectrum. Eqs. (14), (15) and (16) are actually valid even for complex Y and Z in the case of quasi-degenerate neutrino mass spectrum when $\alpha = \beta = 0$. This is because the imaginary parts inside $|\dots|$ and $\text{Re}[\dots]$ in the left- and right-handed sides of Eq. (12) are cancelled out, and only the real parts are constrained as

$$\text{Re}Z = \frac{\kappa - \eta\sqrt{2}}{\eta + \sqrt{2}\kappa} \text{Re}Y \quad (17)$$

with $[\text{Re}Y - \sqrt{2}\text{Re}Z]/\eta > 0$ (< 0) for the case of normal (inverted) mass ordering. Then, it is observed that Eq. (17) is automatically satisfied once Eq. (14) is imposed for complex Y and Z .

In Figs. 3 and 4, the scatter plots of θ_{13} vs θ_{23} are shown for the normal and inverted ordering cases. The black, red and green dots correspond to the cases of $m_{1(3)}^0 = 0$, 0.03 and 0.1 eV, respectively, in the case of normal (inverted) ordering. X and Y are treated as complex parameters with arbitrary phases, while α and β are fixed at zero. It can be seen that the EFM relation becomes clear as the scale of neutrino masses increases.

- Case A-III: $Y = Y^*$ and $Z = Z^*$ in the case of $m_1, m_2 \ll m_3$. Suppose m_1^0 and m_2^0 are negligibly small in comparison with m_3^0 , the dependences of Eq. (12) on the Majorana phases α and β would be obscured. In this case, Eqs. (14), (15) and (16) are again valid for real Y and Z without assuming specific values for α and β .

In Fig. 5, θ_{13} and θ_{23} are numerically computed for the normal ordering case while fixing m_1^0 at zero. α and β run from 0 to 2π , and $|Y|$ is varied from 0 to 0.01 eV (0.003 eV) for the $Z = 0$ ($Z = 2\sqrt{2}Y$) case. In the case of $Z = 2\sqrt{2}Y$, the EFM relation is hazed like Fig. 2, but it is more significant in the present case since

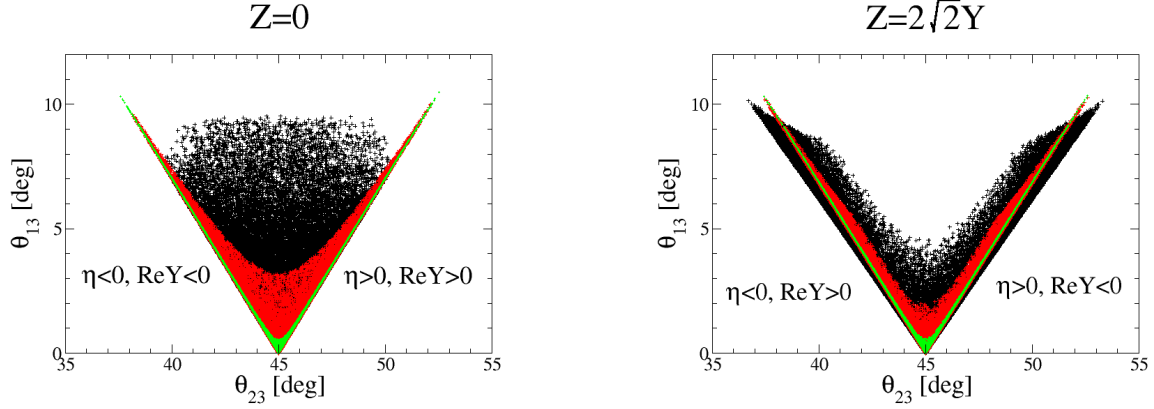


FIG. 3. θ_{13} as a function of θ_{23} for $Z = 0$ (left panel) and $Z = 2\sqrt{2}Y$ (right panel) in the case of normal mass ordering for Case A-II. Δm_{12}^2 , Δm_{23}^2 and θ_{12} are restricted to be within the 1σ bounds. The black, red and green dots correspond to $m_1^0 = 0, 0.03$ and 0.1 eV, respectively.

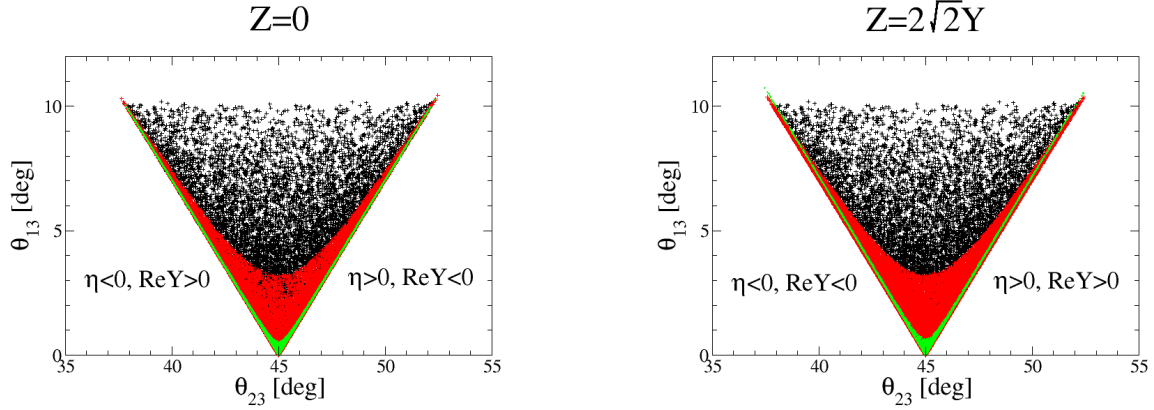


FIG. 4. Legend is the same as Fig. 3, but for the inverted ordering case.

we are looking at a neutrino mass region where the errors of an approximation of $\pm m_1^0 + m_3^0 = \pm m_2^0 + m_3^0$ is maximized.

B. $\eta = 0$

- Case B-I: $\alpha = \beta$ in the case of $m_1 \simeq m_2 \gtrless m_3^0$.

In a case where $m_1^0 = m_2^0$ and $\alpha = \beta$, the right-handed side of Eq. (12) turns out

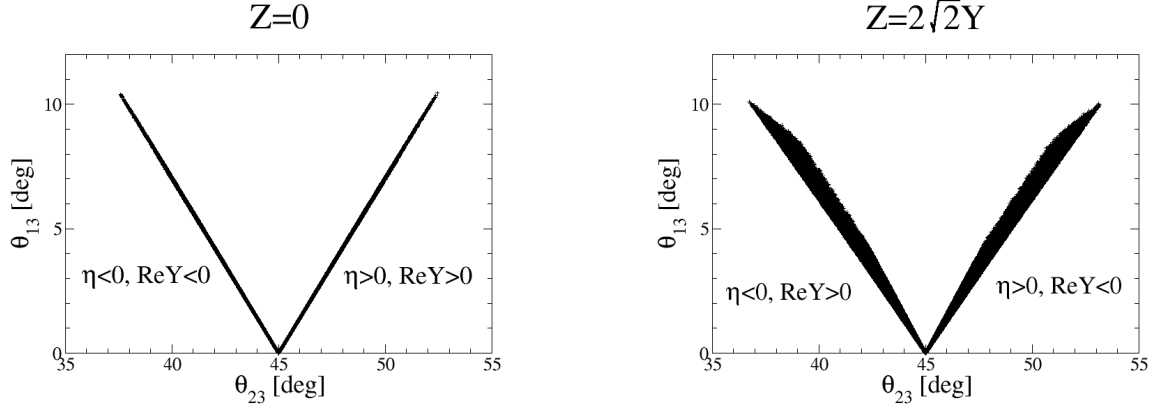


FIG. 5. Legend is the same as Fig. 2, but for Case A-III.

to be

$$\text{Re}[m_1^0 e^{i\alpha}(Y^* - \sqrt{2}Z^*) + m_3^0(Y - \sqrt{2}Z)],$$

and one finds that

$$Z = \frac{Y}{\sqrt{2}} \quad (18)$$

satisfies Eq. (12) for $\eta = 0$, resulting in

$$\delta \bar{M}_\nu = \frac{X}{3\sqrt{2}} \begin{pmatrix} 4 & 1 & 1 \\ 1 & -2 & -2 \\ 1 & -2 & -2 \end{pmatrix} + \frac{\sqrt{3}Y}{2} \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \quad (19)$$

The results of numerical calculations for the case of normal mass ordering are displayed in Fig. 6. In the left panel, m_1^0 is varied within $0.05 \sim 0.1$ eV in which $m_1^0 = m_2^0$ is expected to be reasonable, while m_1^0 is fixed at zero in the right panel in order to see how the stability of θ_{23} is affected by $m_1^0 < m_2^0$. X and Y are treated as complex parameters, and $|Y|$ and $\alpha = \beta$ run from 0 to 0.007 eV and 0 to 2π , respectively, in both cases. It can be seen that θ_{23} scarcely departs from 45° in the left panel, whereas θ_{23} can be $45^\circ \pm 1^\circ$ in the right panel. As for the inverted ordering case, the assumption $m_1^0 = m_2^0$ is always valid, yielding the almost the same figure as that in the left panel of Fig. 6. Hence, we refrain from showing it.

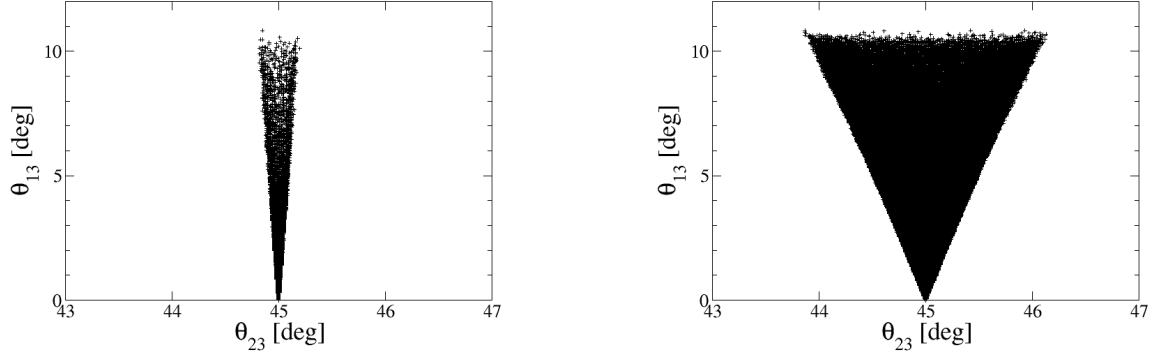


FIG. 6. θ_{13} as a function of θ_{23} in the cases of normal mass ordering for Case B-I. Δm_{12}^2 , Δm_{23}^2 and θ_{12} are restricted to be within the 1σ bounds. In the left panel, m_1^0 is varied within $0.05 \sim 0.1$ eV, while it is fixed at zero in the right panel.

We add a remark that Eq. (18) is equivalent to the condition found in Ref. [6]. It can be seen from Eq. (7) that $Z = Y/\sqrt{2}$ (with $s_{13}^0 = 1/\sqrt{3}$) results in $b = c$; this is what observed in Ref. [6].

- Case B-II: $-Y = Y^*$, $-Z = Z^*$, $\alpha = 0$ or π and $\beta = 0$ or π .

Suppose that Y and Z are pure imaginary and that α and β are equal to 0 or π , the right-handed side of Eq. (12) turns out pure imaginary, and thus Eq. (12) is satisfied for $\eta = 0$.

In Fig. 7, we plot θ_{13} as a function of θ_{23} (left panel) in the case of normal mass ordering for $|Y| = 0 \sim 0.013$ eV, $|Z| = 0 \sim 0.001$ eV and $m_1^0 = 0 \sim 0.1$ eV. This case prefers a somewhat small X . In the right panel, we show the dependence of θ_{23} on $|X|$; one can see that the deviations of θ_{23} from 45° become large as $|X|$ increases. The red line in the left panel corresponds to the case of $X = 0$. Figures for the inverted ordering case are almost the same as Fig. 7, so we refrain from showing them.

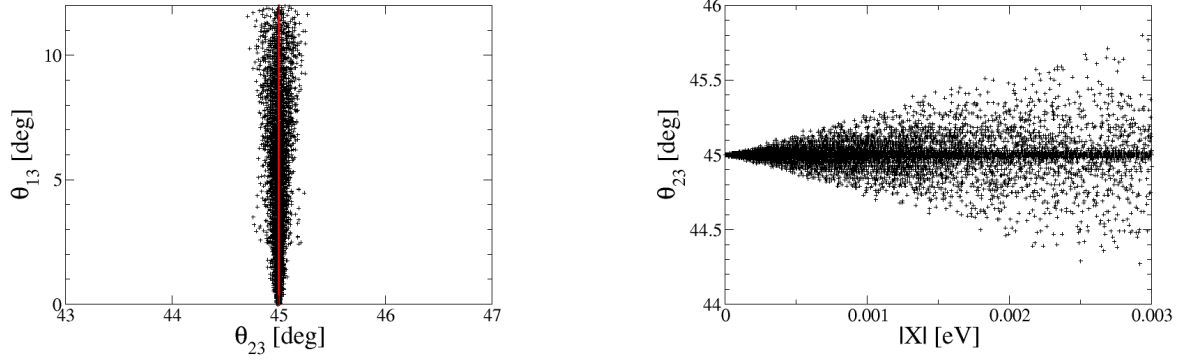


FIG. 7. θ_{13} as a function of θ_{23} (left panel) and θ_{23} as a function of $|X|$ (right panel) in the case of normal mass ordering for Case B-II. Δm_{12}^2 , Δm_{23}^2 and θ_{12} are restricted to be within the 1σ bounds. In the left panel, $|X|$ is varied within $0 \sim 0.001$ eV, and the red line corresponds to the case of $X = 0$.

We mention about the ambiguity of a form of $\delta\bar{M}_\nu$: one can subtract terms proportional to \bar{M}_ν^0 and/or the unit matrix from $\delta\bar{M}_\nu$ without any modification to mixing. For example,

$$\frac{X}{3\sqrt{2}} \begin{pmatrix} 4 & 1 & 1 \\ 1 & -2 & -2 \\ 1 & -2 & -2 \end{pmatrix}$$

can be divided into

$$\frac{X}{3\sqrt{2}} \left[\begin{pmatrix} 3 & 2 & 2 \\ 2 & -3 & -1 \\ 2 & -1 & -3 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \right].$$

The second term does not affect the mixing, and the third term can be embedded into \bar{M}_ν^0 . In this sense

$$\frac{X}{3\sqrt{2}} \begin{pmatrix} 4 & 1 & 1 \\ 1 & -2 & -2 \\ 1 & -2 & -2 \end{pmatrix} \quad \text{and} \quad \frac{X}{3\sqrt{2}} \begin{pmatrix} 3 & 2 & 2 \\ 2 & -3 & -1 \\ 2 & -1 & -3 \end{pmatrix}$$

are equivalent and result in the same mixing. The difference between them is the parametrization of $\delta\bar{M}_\nu$, and it is visible in the basis of $\delta\mathcal{D}_\nu$ in Eq. (6). On one hand, the

original form is translated as

$$(V^0)^\dagger \frac{X}{3\sqrt{2}} \begin{pmatrix} 4 & 1 & 1 \\ 1 & -2 & -2 \\ 1 & -2 & -2 \end{pmatrix} (V^0)^* = X \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

as we defined. On the other hand, the subtracted version turns out

$$(V^0)^\dagger \frac{X}{3\sqrt{2}} \begin{pmatrix} 3 & 2 & 2 \\ 2 & -3 & -1 \\ 2 & -1 & -3 \end{pmatrix} (V^0)^* = X \begin{pmatrix} -\frac{\sqrt{2}}{3} & 1 & 0 \\ 1 & \frac{1}{3\sqrt{2}} & 0 \\ 0 & 0 & -\frac{\sqrt{2}}{3} \end{pmatrix},$$

and non-zero values are entered in the diagonal elements. Namely, only the eigenvalues are altered by this change.

V. POSSIBLE REALIZATIONS OF $\delta\bar{M}_\nu$

We here describe two types of procedures for realizing Eqs. (15), (16) and (19). The first one utilizes a flavor symmetry, while the second one is based on radiative corrections. As for Case B-II, it may be interesting to invoke the timeon mechanism [16], which is a mechanism of spontaneous CP violation triggered by the VEV of a pseudo-scalar.

A. Flavor symmetry approach

We adopt the S_4 symmetry as our flavor symmetry following the definitions given in Ref. [17] and suppose that the three generations of the left- and right-handed leptons belong to $\mathbf{3}_1$ of S_4 . Besides, in order for the S_4 flavor symmetry to be spontaneously broken, the following scalar fields and vacuum expectation values (VEVs) are introduced:

$$\begin{aligned} \langle \phi_{3_1}^\nu \rangle &= (V_a^\nu, V_a^\nu, V_a^\nu) + (t_1^\nu, t_2^\nu, t_3^\nu), \\ \langle \phi_2^\nu \rangle &= (V_b^\nu, V_b^\nu) + (d_1^\nu, d_2^\nu), \\ \langle \phi_{1_1}^\nu \rangle &= V_c^\nu \end{aligned} \tag{20}$$

for the neutrino sector, and

$$\begin{aligned}
\langle \phi_{3_1}^\ell \rangle &= (V_a^\ell, 0, 0) + (t_1^\ell, t_2^\ell, t_3^\ell), \\
\langle \phi_{3_2}^\ell \rangle &= (V_b^\ell, 0, 0) + (u_1^\ell, u_2^\ell, u_3^\ell), \\
\langle \phi_2^\ell \rangle &= (d_1^\ell, d_2^\ell), \\
\langle \phi_{1_1}^\ell \rangle &= V_c^\ell
\end{aligned} \tag{21}$$

for the charged lepton sector. The subscripts beside $\phi^{\nu, \ell}$ indicate representations under the S_4 symmetry, but we do not specify their representations under the gauge symmetries. Here, we are only interested in how the S_4 symmetry is broken by each representation and what sort of breaking terms are arisen, so that we mention nothing about the gauge structures and consider the effective Yukawa interactions

$$\mathcal{L} = \sum_{\mathcal{A}} Y_{\mathcal{A}}^\nu \nu_L^\dagger \nu_L^* \phi_{\mathcal{A}}^\nu + \sum_{\mathcal{A}} Y_{\mathcal{A}}^\ell \bar{e}_L e_R \phi_{\mathcal{A}}^\ell + h.c. , \tag{22}$$

where ν_L is the left-handed neutrino, e_L and e_R are the left- and right-handed charged leptons, respectively, and $Y_{\mathcal{A}}^{\nu, \ell}$ stands for a Yukawa coupling. For example, one can imagine $Y_{3_1}^\ell \bar{L} \phi_{3_1}^\ell e_R$ if $\phi_{3_1}^\ell$ is $SU(2)_L$ doublet. Meanwhile, $(f/\Lambda^3) LLHH(\phi_{3_1}^\nu \phi_{3_1}^\nu)_2$ could be invented if $\phi_{3_1}^\nu$ is a gauge singlet scalar, where L is the left-handed doublet lepton, H is the SM Higgs, f is a dimension-less coupling and Λ denotes a cut-off scale. In this case, $(\phi_{3_1}^\nu \phi_{3_1}^\nu)_2/\Lambda$ corresponds to ϕ_2^ν and $f\langle H \rangle^2/\Lambda^2 = Y_2^\nu$ in Eq. (22).

Note that, in the neutrino sector, we have omitted $\phi_{1_2}^\nu$ and $\phi_{3_2}^\nu$ since $\mathbf{1}_2$ and $\mathbf{3}_2$ are not produced from $\nu_L^\dagger(\mathbf{3}_1) \otimes \nu_L^*(\mathbf{3}_1)$. Similarly, $\phi_{1_2}^\ell$ have been omitted in the charged lepton sector.

We presume that $V_{a,b,c}^\nu$ and $V_{a,b,c}^\ell$ are sufficiently larger than the other VEVs and that they constitute the leading-order mass matrices:

$$M_\nu^0 = \begin{pmatrix} 2V_a^\nu + V_c^\nu & -V_a^\nu + V_b^\nu & -V_a^\nu + V_b^\nu \\ -V_a^\nu + V_b^\nu & 2V_a^\nu + V_b^\nu & -V_a^\nu + V_c^\nu \\ -V_a^\nu + V_b^\nu & -V_a^\nu + V_c^\nu & 2V_a^\nu + V_b^\nu \end{pmatrix} \tag{23}$$

and

$$M_\ell^0 = \begin{pmatrix} 2V_a^\ell + V_c^\ell & 0 & 0 \\ 0 & 0 & -V_a^\ell + V_b^\ell + V_c^\ell \\ 0 & -V_a^\ell - V_b^\ell + V_c^\ell & 0 \end{pmatrix}, \tag{24}$$

where the Yukawa couplings $Y_{\mathcal{A}}^\nu$ and $Y_{\mathcal{A}}^\ell$ are embedded into the VEVs. The same shall apply in the following discussions. On one hand, M_ℓ^0 is diagonalized by rotations of the right-handed fields, resulting in $m_e^0 = 2V_a^\ell + V_c^\ell$, $m_\mu^0 = -V_a^\ell + V_b^\ell + V_c^\ell$ and $m_\tau^0 = -V_a^\ell - V_b^\ell + V_c^\ell$ as well as $U_\ell^0 = 1$. On the other hand, by comparing M_ν^0 with Eq. (5), one immediately observes that M_ν^0 takes the same form as \bar{M}_ν^0 for $s_{12}^0 = 1/\sqrt{3}$, i.e. $V_a^\nu = \frac{1}{6}[m_1^0 e^{i\alpha} + m_3^0]$, $V_b^\nu = \frac{1}{6}[-m_1^0 e^{i\alpha} + 2m_2^0 e^{i\beta} + m_3^0]$ and $V_c^\nu = \frac{1}{3}[m_1^0 e^{i\alpha} + m_2^0 e^{i\beta} - m_3^0]$. Thus, these mass matrices lead to the TBM mixing, and we identify the above M_ν^0 with \bar{M}_ν^0 . As it is well known [18], the mass matrices possess residual $\mathbb{Z}_2 \times \mathbb{Z}_2'$ and \mathbb{Z}_3 symmetries in the neutrino and charged lepton sectors, respectively. They are the symmetries ensuring the TBM mixing.

The other VEVs break the residual symmetries and yield the correction term $\delta\bar{M}_\nu$. In the charged lepton sector, we have

$$\delta M_\ell = \begin{pmatrix} 0 & d_1^\ell & d_2^\ell \\ d_1^\ell & d_2^\ell & 0 \\ d_2^\ell & 0 & d_1^\ell \end{pmatrix} + \begin{pmatrix} 2t_1^\ell & -t_3^\ell & -t_2^\ell \\ -t_3^\ell & 2t_2^\ell & -t_1^\ell \\ -t_2^\ell & -t_1^\ell & 2t_3^\ell \end{pmatrix} + \begin{pmatrix} 0 & u_3^\ell & -u_2^\ell \\ -u_3^\ell & 0 & u_1^\ell \\ u_2^\ell & -u_1^\ell & 0 \end{pmatrix}. \quad (25)$$

Let us convert them from the charged lepton sector into the neutrino sector along the procedure outlined in Sec. II. In the limit of $m_e^0 = 0$ and $|\delta M_\ell|^2 = 0$, $M_\ell M_\ell^\dagger$ is given by

$$M_\ell M_\ell^\dagger = \begin{pmatrix} 0 & 0 & 0 \\ 0 & |m_\mu^0|^2 & 0 \\ 0 & 0 & |m_\tau^0|^2 \end{pmatrix} + \begin{pmatrix} 0 & (m_\mu^0)^*(d_2^\ell - t_2^\ell - u_2^\ell) & 0 \\ m_\mu^0(d_2^\ell - t_2^\ell - u_2^\ell)^* & -2\text{Re}[m_\mu^0(t_1^\ell)^*] + 2\text{Re}[m_\mu^0(u_1^\ell)^*] & m_\mu^0(d_1^\ell + 2t_3^\ell)^* \\ 0 & (m_\mu^0)^*(d_1^\ell + 2t_3^\ell) & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & (m_\tau^0)^*(d_1^\ell - t_3^\ell + u_3^\ell) \\ 0 & 0 & (m_\tau^0)^*(d_2^\ell + 2t_2^\ell) \\ m_\tau^0(d_1^\ell - t_3^\ell + u_3^\ell)^* & m_\tau^0(d_2^\ell + 2t_2^\ell)^* & -2\text{Re}[m_\tau^0(t_1^\ell)^*] - 2\text{Re}[m_\tau^0(u_1^\ell)^*] \end{pmatrix}.$$

By regarding the second and third terms as small perturbations to the first term, δU_ℓ is

found to be

$$\delta U_\ell \simeq \begin{pmatrix} 0 & \frac{d_2^\ell - t_2^\ell - u_2^\ell}{m_\mu^0} & \frac{d_1^\ell - t_3^\ell + u_3^\ell}{m_\tau^0} \\ -\left(\frac{d_2^\ell - t_2^\ell - u_2^\ell}{m_\mu^0}\right)^* & 0 & \frac{d_2^\ell + 2t_2^\ell}{m_\tau^0} \\ -\left(\frac{d_1^\ell - t_3^\ell + u_3^\ell}{m_\tau^0}\right)^* & -\left(\frac{d_2^\ell + 2t_2^\ell}{m_\tau^0}\right)^* & 0 \end{pmatrix},$$

where we have dropped terms proportional to $m_\mu^0/(m_\tau^0)^2$ in the 23 and 32 elements. Now we are ready to calculate $\delta U_\ell^\dagger M_\nu^0 (U_\ell^0)^* + (U_\ell^0)^\dagger M_\nu^0 \delta U_\ell^*$, but the expression is long and very complicated. Hence, we here suppose CP invariance in the charged lepton sector; then it is simplified to be

$$\begin{aligned} & \delta U_\ell^\dagger M_\nu^0 (U_\ell^0)^* + (U_\ell^0)^\dagger M_\nu^0 \delta U_\ell^* = \\ & \frac{\mathcal{V}_{12}^\ell}{m_\mu^0} \begin{pmatrix} -2(-V_a^\nu + V_b^\nu) & * & * \\ (-V_a^\nu + V_c^\nu) - (-V_a^\nu + V_b^\nu) & 2(-V_a^\nu + V_b^\nu) & * \\ -(-V_a^\nu + V_c^\nu) & (-V_a^\nu + V_b^\nu) & 0 \end{pmatrix} \\ & + \frac{\mathcal{V}_{23}^\ell}{m_\tau^0} \begin{pmatrix} 0 & * & * \\ -(-V_a^\nu + V_b^\nu) & -2(-V_a^\nu + V_c^\nu) & * \\ (-V_a^\nu + V_b^\nu) & 0 & 2(-V_a^\nu + V_c^\nu) \end{pmatrix} \\ & + \frac{\mathcal{V}_{13}^\ell}{m_\tau^0} \begin{pmatrix} -2(-V_a^\nu + V_b^\nu) & * & * \\ -(-V_a^\nu + V_c^\nu) & 0 & * \\ (-V_a^\nu + V_c^\nu) - (-V_a^\nu + V_b^\nu) & (-V_a^\nu + V_b^\nu) & 2(-V_a^\nu + V_b^\nu) \end{pmatrix}, \quad (26) \end{aligned}$$

where $\mathcal{V}_{12}^\ell = d_2^\ell - t_2^\ell - u_2^\ell$, $\mathcal{V}_{23}^\ell = d_2^\ell + 2t_2^\ell$ and $\mathcal{V}_{13}^\ell = d_1^\ell - t_3^\ell + u_3^\ell$. (t_1^ℓ and u_1^ℓ only contribute to the charged lepton masses.)

Corrections from the neutrino sector are simply given by

$$(U_\ell^0)^\dagger \delta M_\nu (U_\ell^0)^* = \begin{pmatrix} 0 & d_1^\nu & d_2^\nu \\ d_1^\nu & d_2^\nu & 0 \\ d_2^\nu & 0 & d_1^\nu \end{pmatrix} + \begin{pmatrix} 2t_1^\nu & -t_3^\nu & -t_2^\nu \\ -t_3^\nu & 2t_2^\nu & -t_1^\nu \\ -t_2^\nu & -t_1^\nu & 2t_3^\nu \end{pmatrix}. \quad (27)$$

In summary, $\delta \bar{M}_\nu$ consists of Eqs. (26) and (27) in this scenario. For instance, in a case where $\alpha = \beta$, the first term of Eq. (26) turns out to be the second term of Eq. (19) since $-V_a^\nu + V_b^\nu$ is much smaller than $-V_a^\nu + V_c^\nu$ in this case. Similarly, the second term of Eq. (16) can be obtained by combining the second and third terms of Eq. (26) when $\mathcal{V}_{13}^\ell = 2\mathcal{V}_{23}^\ell$. As for Eq. (27), $(t_1^\nu, t_2^\nu, t_3^\nu) \propto (2, -1, -1)$ and $(d_1^\nu, d_2^\nu) \propto (1, -1)$ yield the first and second terms of Eq. (15), respectively.

B. Radiative corrections

We consider the combination of tree-level and one-loop neutrino masses. The tree-level neutrino mass matrix is assumed to take the form of \bar{M}_ν^0 in the diagonal basis of the charged lepton mass matrix, leading to the TBM mixing at tree level. For this setup, we insert two types of one-loop radiative corrections and show that they may be useful to realize a preferable $\delta\bar{M}_\nu$.

The first example is proposed in Ref. [19], in which we have

$$\delta\bar{M}_\nu = \frac{\bar{M}_\nu^0 D_\ell^2 + D_\ell^2 \bar{M}_\nu^0}{v_{\text{ew}}^2} \times I^{\text{loop}} \quad (28)$$

at one-loop level, where $D_\ell = \text{Diag}(m_e, m_\mu, m_\tau)$, $v_{\text{ew}} = 174$ GeV and I^{loop} is a dimensionless parameter including some loop factors. The details of I^{loop} depend on model details, so we remain it arbitrary in this study. If $\alpha = 0$ and $\beta = \pi$ in the case of quasi-degenerate mass spectrum, Eq. (28) turns out to be

$$\delta\bar{M}_\nu \simeq \frac{m_\nu^0}{3} \left[m_\mu^2 \begin{pmatrix} 0 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & 0 \end{pmatrix} + m_\tau^2 \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & -2 \\ -2 & -2 & 2 \end{pmatrix} \right], \quad (29)$$

where $m_\nu^0 \equiv m_1^0 \simeq m_2^0 \simeq m_3^0$, and we have omitted the term proportional to m_e^2 . Then, by adding

$$- \frac{m_\nu^0}{3} (m_\mu^2 + m_\tau^2) \begin{pmatrix} 0 & -1 & -1 \\ -1 & 1 & -2 \\ -1 & -2 & 1 \end{pmatrix}$$

which can be absorbed into \bar{M}_ν^0 , one obtains the second term of Eq. (15):

$$\delta\bar{M}_\nu = \frac{m_\nu^0}{3} (m_\mu^2 - m_\tau^2) \begin{pmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}. \quad (30)$$

This example may be applicable to Case A-II⁴.

⁴ Recall that there is no constraint on β in the case of $Z = 0$.

The second example is the so-called Zee-model given in Ref. [20], which induces

$$\delta\bar{M}_\nu = \begin{pmatrix} 0 & f_{e\mu}(m_\mu^2 - m_e^2) & f_{e\tau}(m_\tau^2 - m_e^2) \\ f_{e\mu}(m_\mu^2 - m_e^2) & 0 & f_{\mu\tau}(m_\tau^2 - m_\mu^2) \\ f_{e\tau}(m_\tau^2 - m_e^2) & f_{\mu\tau}(m_\tau^2 - m_\mu^2) & 0 \end{pmatrix} \frac{1}{\Gamma}$$

$$\equiv \begin{pmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{pmatrix}$$

at one-loop level. $f_{\alpha\beta}$ are Yukawa couplings, and Γ contains loop factors and possesses mass-dimension one. Similar to the first example, by adding and subtracting terms proportional to

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

one arrives at Eq. (19):

$$\delta\bar{M}_\nu = X \begin{pmatrix} 4 & 1 & 1 \\ 1 & -2 & -2 \\ 1 & -2 & -2 \end{pmatrix} + Y \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad (31)$$

with $X = (a + b - 2c)/18$ and $Y = (a - b)/2$.

VI. SUMMARY

The TBM mixing may still be applicable as a leading order one in the presence of small corrections, although the exact TBM mixing was ruled out by the discovery of a non-zero θ_{13} . If this direction is adopted, one immediately faces with a difficult challenge of revealing structures and origins of the corrections. In this work, employing Eq. (1) as a guide, we seek for preferable correction terms to the charged lepton and neutrino mass matrices and discuss their origins. The latest global fits of neutrino oscillation parameters point to $\eta = \pm 1/\sqrt{2}$, whereas the results of the ν_μ disappearance mode seem to prefer $\eta = 0$. We have succeeded in deriving the general condition Eq. (12) for obtaining Eq. (1), but it has also been found that the condition is complicated by the neutrino masses and

CP violating phases. Hence, in order to find out simple and realizable correction terms, we concentrate on a specific neutrino mass spectrum, e.g. the quasi-degenerate spectrum, and/or characteristic CP violating phases, e.g. vanishing Majorana phases. Under such simplified environments, we arrive at Eqs. (15) and (16) for the case of $\eta = \pm 1/\sqrt{2}$, while Eq. (19) for $\eta = 0$. Furthermore, we develop several ways to realize the obtained correction terms by utilizing the spontaneous breaking of an S_4 flavor symmetry or one-loop radiative corrections.

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